Finite Automata (FA) and Monadic Second Order logic (MSO).

- FA: executable model with good (decidable) properties.
- MSO (over words): very expressive and yet simple logic.
- Both equally expressive over words and trees (Büchi).

  ⇢ Qualitative properties over words.

Quantitative properties are also important (today).

Example

- number of -symbols.
- length of the largest sequence of -symbols.

How can we extend finite automata or MSO
to define these properties (or functions)?

Weighted automata

General automata framework to define quantitative properties over words.

- (Boolean) automata,
- Probabilistic automata,
- Distance automata,
- Multiplicity automata, etc...

Extension of finite automata with weights from a fix semiring.

Semiring (reminder)

Definition

A (commutative) semiring is an algebraic structure $\mathbb{S} = (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ where:

- $(S, \oplus, \mathbb{0})$ and $(S, \odot, \mathbb{1})$ are commutative monoids,
- multiplication distributes over addition, and
- $\mathbb{0} \odot s = s \odot \mathbb{0} = \mathbb{0}$ for each $s \in S$.

Example

- Natural numbers: $(\mathbb{N}, +, \cdot, 0, 1)$.
- Boolean: $(\{\text{true}, \text{false}\}, \vee, \wedge, \text{false}, \text{true})$.
- Min-plus: $(\mathbb{N}_\infty, \min, +, \infty, 0)$.
- Max-plus: $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$.

Weighted automata (definition)

Fix a semiring $\mathbb{S}$ and a finite alphabet Γ.

Definition

A weighted automata over $\mathbb{S}$ and Γ is a tuple $\mathcal{A} = (\Gamma, \mathbb{S}, Q, E, I, F)$:

- $E : Q \times \Gamma \times Q \to S$ is the transition relation ($p \xrightarrow{a/s} q$), and
- $I, F : Q \to S$ is the initial and final function.

Semantics

- A run ρ of $\mathcal{A}$ over $a_1 \ldots a_n \in \Gamma^*$ is:

$$\rho = q_0 \xrightarrow{a_1/s_1} q_1 \xrightarrow{a_2/s_2} \cdots \xrightarrow{a_n/s_n} q_n$$

- The weight of run ρ of $\mathcal{A}$:

$$\text{weight}(\rho) = I(q_0) \odot \bigodot_{i=1}^{n} s_i \odot F(q_n)$$

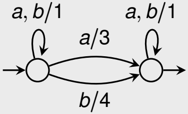
- $\mathcal{A}$ defines the function $[\![\mathcal{A}]\!] : \Gamma^* \to S$:

$$[\![\mathcal{A}]\!](w) = \bigoplus_{\rho \in \text{Run}_{\mathcal{A}}(w)} \text{weight}(\rho)$$

Weighted automata (examples)

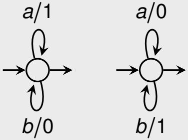
Over $(\mathbb{N}, +, \cdot, 0, 1)$

- $f(w) = 3 \cdot |w|_a + 4 \cdot |w|_b$



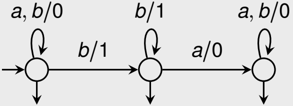
Over $(\mathbb{N}_\infty, \min, +, \infty, 0)$

- $f(w) = \min\{|w|_a, |w|_b\}$



Over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

- $f(w)$ = maximum length of all infix sequences of b's



What is a good logic to define quantitative properties?

Weighted MSO (Droste & Gastin 2005)

Disadvantages:

- Semantical definition of valid formulas.
- Inherits the undecidability results of weighted automata.

We want a quantitative logic that:

1. has a simple and purely syntactical definition,
2. as expressive as weighted automata, and
3. with good decidability properties.

We propose:

Quantitative Monadic Second Order Logic (QMSO)

1. General framework for adding quantitative properties to any boolean logic.
2. Subfragments of QMSO capture different subclasses of WA.
3. Subfragments of QMSO with good decidability properties.

More results in the paper:

- Evalution of QMSO with respect to counting complexity classes.

Quantitative Monadic Second-Order Logic

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LICS 2013

Outline

QMSO and WA

QMSO and subclasses of WA

Beyond WA

Conclusions

Quantitative Monadic Second Order Logic (QMSO)

For each $w \in \Gamma^*$, we represent $w := (\{1, \ldots, |w|\}, \leq, \{P_a\}_{a \in \Gamma})$.

Syntax of QMSO$[\mathbb{S}, \Gamma]$

$$\varphi \quad := \quad P_a(x) \mid x \leq y \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\, \varphi \mid \exists X.\, \varphi$$

$$\theta \quad := \quad \varphi \mid s \in S \mid \theta \oplus \theta \mid \theta \odot \theta \mid \Sigma x.\, \theta \mid \Pi x.\, \theta \mid \Sigma X.\, \theta$$

Semantic of QMSO$[\mathbb{S}, \Gamma]$

$$[\![\varphi]\!](w, \sigma) \quad := \quad \begin{cases} \mathbb{1} & \text{if } (w, \sigma) \models \varphi \\ \mathbb{0} & \text{otherwise} \end{cases}$$

$$[\![s]\!](w, \sigma) \quad := \quad s$$

$$[\![\theta_1 \oplus \theta_2]\!](w, \sigma) \quad := \quad [\![\theta_1]\!](w, \sigma) \oplus [\![\theta_2]\!](w, \sigma)$$

$$[\![\Pi x.\, \theta(x)]\!](w, \sigma) \quad := \quad \bigodot_{i \in \mathrm{dom}(w)} [\![\theta(x)]\!](w, \sigma[x \to i])$$

$$[\![\Sigma X.\, \theta(X)]\!](w, \sigma) \quad := \quad \bigoplus_{I \subseteq \mathrm{dom}(w)} [\![\theta(X)]\!](w, \sigma[X \to I])$$

Quantitative Monadic Second Order Logic (QMSO)

The syntax of $\mathrm{QMSO}[\mathbb{S}, \Gamma]$ depends on the semiring.

Syntax of $\mathrm{QMSO}[(\mathbb{N}, +, \cdot, 0, 1), \Gamma]$

$$\varphi \quad := \quad P_a(x) \mid x \le y \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\, \varphi \mid \exists X.\, \varphi$$

$$\theta \quad := \quad \varphi \mid s \in \mathbb{N} \mid \theta + \theta \mid \theta \cdot \theta \mid \Sigma x.\, \theta \mid \Pi x.\, \theta \mid \Sigma X.\, \theta$$

Syntax of $\mathrm{QMSO}[(\mathbb{N}_\infty, \min, +, \infty, 0), \Gamma]$

$$\varphi \quad := \quad P_a(x) \mid x \le y \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\, \varphi \mid \exists X.\, \varphi$$

$$\theta \quad := \quad \varphi \mid s \in \mathbb{N}_\infty \mid \min\{\theta, \theta\} \mid \theta + \theta \mid \mathrm{Min}\, x.\, \theta \mid \Sigma x.\, \theta \mid \mathrm{Min}\, X.\, \theta$$

Syntax of $\mathrm{QMSO}[(\mathbb{N}_{-\infty}, \max, +, -\infty, 0), \Gamma]$

$$\varphi \quad := \quad P_a(x) \mid x \le y \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\, \varphi \mid \exists X.\, \varphi$$

$$\theta \quad := \quad \varphi \mid s \in \mathbb{N}_{-\infty} \mid \max\{\theta, \theta\} \mid \theta + \theta \mid \mathrm{Max}\, x.\, \theta \mid \Sigma x.\, \theta \mid \mathrm{Max}\, X.\, \theta$$

Examples of QMSO formulas

Over $(\mathbb{N}, +, \cdot, 0, 1)$

- $f(w) = 3 \cdot |w|_a + 4 \cdot |w|_b$

$$\Sigma x. \left(3 \cdot P_a(x) + 4 \cdot P_b(x) \right)$$

Over $(\mathbb{N}_\infty, \min, +, \infty, 0)$

- $f(w) = \min\{|w|_a, |w|_b\}$

$$\min \left\{ \Sigma x. P_a(x) \mapsto 1 \;,\; \Sigma x. P_b(x) \mapsto 1 \right\}$$

where $P_a(x) \mapsto 1 \;:=\; \min\{ P_a(x) + 1, \neg P_a(x) \}.$

Over $(\mathbb{N}_{-\infty}, \max, +, -\infty, 0)$

- $f(w)$ = maximum length of all infix sequences of b's

$$\mathrm{Max}\, x. \left(\Sigma y.\ \mathrm{interval}_b(x, y) \mapsto 1\right)$$

where $\mathrm{interval}_b(x, y) \;:=\; x \leq y \wedge \forall z. (x \leq z \wedge z \leq y) \rightarrow P_b(z).$

Subfragments of QMSO

1. QMSO(Op) restricted to operators $\mathrm{Op} \subseteq \{\oplus, \odot, \Sigma_x, \Pi_x, \Sigma_X\}$.

$$
\begin{array}{rcl}
\oplus & = & \text{semiring addition} \\
\odot & = & \text{semiring multiplication} \\
\Sigma_x & = & \text{first-order addition} \\
\Pi_x & = & \text{first-order multiplication} \\
\Sigma_X & = & \text{second-order addition}
\end{array}
$$

Example

$$\text{Full QMSO} := \text{QMSO}(\Sigma_X, \Pi_x, \Sigma_x, \oplus, \odot)$$

Subfragments of QMSO

1. QMSO(Op) restricted to operators Op $\subseteq \{\oplus, \odot, \Sigma_x, \Pi_x, \Sigma_X\}$.

2. Alternation and Nesting of semiring quantifiers.

Example

- QMSO($\Sigma_X \Sigma_x \Pi_x, \oplus, \odot$):

$$\Sigma X.\ \big(\Sigma y.\, \Pi z.\, \varphi(X, z)\big) \ \oplus\ \big(\Pi z_1.\, \Pi z_2.\, \theta(X, z_1, z_2)\big)$$

- QMSO($\Sigma_x \Pi_x^1, \oplus, \odot$):

$$\Sigma x.\ \big(\Sigma y.\, \Pi z.\, \varphi(x, y, z)\big) \ \odot\ \big(\Pi z.\, \theta(x, z)\big)$$

- QMSO($\Pi_x^n, \oplus, \odot$), $n \in \mathbb{N}$:

$$\Pi x_1. \cdots \Pi x_n.\, \theta(x_1, \ldots, x_n)$$

QMSO and weighted automata

QMSO is too expressive to capture weighted automata!

Over $(\mathbb{N}, +, \cdot, 0, 1)$

- $[\![\Pi x.\, \Pi y.\, 2]\!](w) = 2^{|w|^2}$.
- For every weighted automata $\mathcal{A}$ over $(\mathbb{N}, +, \cdot, 0, 1)$:

$$[\![\mathcal{A}]\!](w) \in 2^{O(|w|)}$$

QMSO and weighted automata

QMSO is too expressive to capture weighted automata!

Definition

Quantitative Iteration Logic (QIL) $:=$ $\text{QMSO}(\Sigma_{X,x}\Pi^1_x, \oplus, \odot)$.

Theorem

A function $f : \Gamma^* \to \mathbb{S}$ is definable by a weighted automaton over $\mathbb{S}$ and Γ if, and only if, f is definable by a formula in $\text{QIL}[\mathbb{S}, \Gamma]$.

$$\text{Weighted Automata} \equiv \text{QIL}\,.$$

Undecidable properties of QIL

Quantitative generalization of classical decision problems:

- Equivalence: $[\![\theta_1]\!](w) = [\![\theta_2]\!](w)$ for all $w \in \Gamma^*$,
- Containment: $[\![\theta_1]\!](w) \leq [\![\theta_2]\!](w)$ for all $w \in \Gamma^*$.

Proposition

The following problems are undecidable:

1. Containment of formulas in $\mathrm{QMSO}(\Sigma_x \Pi^1_x, \oplus, \odot)$ over $(\mathbb{N}, +, \cdot, 0, 1)$.
2. Equivalence and containment of formulas in $\mathrm{QMSO}(\Sigma_x \Pi^1_x, \oplus, \odot)$ over $(\mathbb{N}_\infty, \min, +, \infty, 0)$.

Outline

QMSO and WA

QMSO and subclasses of WA

Beyond WA

Conclusions

Different fragments of QMSO captures different subclasses of WA

Classes of Weighted Automata (WA) depending on the ambiguity:

- Deterministic WA (DWA).
- Unambiguous WA (*unamb*- WA):

$$|\mathrm{Run}_{\mathcal{A}}(w)| \leq 1 \text{ for all } w \in \Sigma^*$$

- Finite Ambiguous WA (*fin*- WA):

$$|\mathrm{Run}_{\mathcal{A}}(w)| < k \text{ for all } w \in \Sigma^*$$

- Polynomially Ambiguous WA (*poly*- WA):

$$|\mathrm{Run}_{\mathcal{A}}(w)| \in O(|w|^k)$$

$$\text{DWA} \subsetneq \textit{unamb}\text{- WA} \subsetneq \textit{fin}\text{- WA} \subsetneq \textit{poly}\text{- WA} \subsetneq \text{WA}$$

Unambiguous and finitely ambiguous weighted automata are captured by QMSO

Subfragment $\mathrm{QMSO}(\mathrm{Op}, \oplus_b)$: $\oplus$-operator is restricted to a "*base*" level.

Example

$$\big(\Pi x.\, P_a(x) \oplus P_b(x)\big) \odot \big(\Pi x.\, \exists z.\, x \leq z \wedge P_a(z)\big) \quad \in \quad \mathrm{QMSO}(\Pi^1_x, \oplus_b, \odot)$$

$$\big(\Pi x.\, P_a(x) \oplus P_b(x)\big) \oplus \big(\Pi x.\, \exists z.\, x \leq z \wedge P_a(z)\big) \quad \notin \quad \mathrm{QMSO}(\Pi^1_x, \oplus_b, \odot)$$

Unambiguous and finitely ambiguous weighted automata are captured by QMSO

Subfragment $\mathrm{QMSO}(\mathrm{Op}, \oplus_b)$: $\oplus$-operator is restricted to a "*base*" level.

Theorem

$$\begin{aligned} \textit{unamb-}\,\mathrm{WA} &\equiv \mathrm{QMSO}(\Pi^1_x, \oplus_b, \odot) \\ \textit{fin-}\,\mathrm{WA} &\equiv \mathrm{QMSO}(\Pi^1_x, \oplus, \odot) \end{aligned}$$

Proof idea.

From $\mathrm{QMSO}(\Pi^1_x, \oplus_b, \odot)$ to *unamb-* WA:

- Exploit unambiguity to express formulas of the form $\Pi x.\ \bigoplus_{i \in I} \bigodot_{j \in J} \varphi_{i,j}(x)$.

From $\mathrm{QMSO}(\Pi^1_x, \oplus, \odot)$ to *fin-* WA:

- Use *disambiguation* theorem presented in Klimann et all, 2004.

Polynomial ambiguous weighted automata are also captured by QMSO

Theorem

$$\textit{poly}\text{-}\mathrm{WA} \equiv \mathrm{QMSO}(\Sigma_x \Pi_x^1, \oplus, \odot)$$

Proof idea.

From *poly*-WA to $\mathrm{QMSO}(\Sigma_x \Pi_x^1, \oplus, \odot)$:

- Exploit structural properties of the components of a *poly*-WA.

Which fragment captures deterministic weighted automata?

The *forward-iterator* $(\cdot)^{\rightarrow}$ and the *backward-iterator* $(\cdot)^{\leftarrow}$

$$\llbracket \theta^{\rightarrow} \rrbracket(w,\sigma) = \bigodot_{i=1}^{n} \llbracket \theta \rrbracket(w[1..i],\sigma)$$

$$\llbracket \theta^{\leftarrow} \rrbracket(w,\sigma) = \bigodot_{i=1}^{n} \llbracket \theta \rrbracket(w[i..n],\sigma)$$

Over $(\mathbb{N}_\infty, \min, +, \infty, 0)$

- $f(w)$ = number of prefixes of w that satisfy φ.

$$(\min\{\ \varphi + 1,\ \neg\varphi\ \})^{\rightarrow}.$$

Which fragment captures deterministic weighted automata?

The *forward-iterator* $(\cdot)^{\rightarrow}$ and the *backward-iterator* $(\cdot)^{\leftarrow}$

$$\llbracket \theta^{\rightarrow} \rrbracket(w,\sigma) = \bigodot_{i=1}^{n} \llbracket \theta \rrbracket(w[1..i],\sigma)$$

$$\llbracket \theta^{\leftarrow} \rrbracket(w,\sigma) = \bigodot_{i=1}^{n} \llbracket \theta \rrbracket(w[i..n],\sigma)$$

Theorem

$$\text{DWA} \equiv \text{QMSO}(^{\rightarrow}, \oplus_b, \odot)$$

$$\textit{co-}\,\text{DWA} \equiv \text{QMSO}(^{\leftarrow}, \oplus_b, \odot)$$

* the $(\cdot)^{\rightarrow}$- and $(\cdot)^{\leftarrow}$-operator cannot be nested.

Connection of determinization of WA with logic.

Fragments with good decidability properties

Corollary

The following problems are decidable:

1. *Equivalence and containment problem of formulas in* $\mathrm{QMSO}(\Pi^1_x, \oplus_b, \odot)$ *over* $(\mathbb{N}, +, \cdot, 0, 1)$.
2. *Equivalence and containment problem of formulas in* $\mathrm{QMSO}(\Pi^1_x, \oplus, \odot)$ *over* $(\mathbb{N}_\infty, \min, +, \infty, 0)$.

$\mathrm{QMSO}(\Pi^1_x, \oplus_b, \odot)$ and $\mathrm{QMSO}(\Pi^1_x, \oplus, \odot)$ are good fragments.

Outline

How to go further from these (good) fragments?

1. Additive fragment: $\mathrm{QMSO}(\Sigma_x^k \Pi_x^1, \oplus, \odot_b)$.

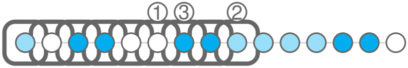
Theorem

For all $k \in \mathbb{N}$: $\quad poly^k\text{-WA} \equiv \mathrm{QMSO}(\Sigma_x^k \Pi_x^1, \oplus, \odot_b)$.

2. Multiplicative fragment: $\mathrm{QMSO}(\Pi_x^k, \oplus_b, \odot)$.

Two-way weighted automata with nested pebbles.

Two-way weighted automata with nested pebbles



In the boolean case:

Two-way weighted automata with nested pebbles ≡ regular languages

Different subclasses of 2WA:

- Two-way WA with *k*-nested pebbles (2WA-k).
- Deterministic 2WA-k (2DWA-k).
- Unambiguous 2WA-k (*unamb-* 2WA-k).

Multiplicative fragment and two-way WA with nested pebbles

Theorem

The following classes of WA and subfragments of QMSO are equally expressive over Γ and $\mathbb{S}$:

1. 2DWA-0,
2. *unamb-* 2WA-0,
3. *unamb-* WA, and
4. $\mathrm{QMSO}(\Pi_x^1, \oplus_b, \odot)$.

Theorem

For every $k \in \mathbb{N}$, there exists an effective translation between the following classes of WA and subfragments of QMSO over Γ and $\mathbb{S}$:

1. 2DWA-k,
2. *unamb-* 2WA-k, and
3. $\mathrm{QMSO}(\Pi_x^{k+1}, \oplus_b, \odot)$.

Outline

Future work

Logic-side:

- Relation between (inner) boolean logic and semiring operators.
- Expressibility of QMSO over more general structures.

Automata-side:

- Decidability properties of subclasses of WA motivated by QMSO.
- Determinization of WA.