Descriptive complexity for counting complexity classes

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Descriptive complexity has been very fruitful in connecting **logics** with **computational complexity**

$$
\begin{array}{ccc}
\mathrm{NP} & \equiv & \exists\mathsf{SO} \\
\mathrm{coNP} & \equiv & \forall\mathsf{SO} \\
\mathrm{P} & \equiv & \mathsf{LFP}_{\leq} \\
\mathrm{NL} & \equiv & \mathsf{TC}_{\leq} \\
\mathrm{AC}_0 & \equiv & \mathsf{FO} + \mathtt{Bit} \\
\mathrm{PSPACE} & \equiv & \mathsf{PFP}_{\leq} \\
\vdots & \vdots & \vdots
\end{array}
$$

Many **applications** in diverse areas like:

1. Computational complexity and logics.
2. Database management systems.
3. Verification systems.

. . . but computational complexity
is not only about `true` or `false`

One would like to study the **complexity** of problems like:

"How many valuations satisfies my boolean formula?"

"How many simple paths
are connecting two vertices in my graph?"

...but computational complexity is not only about `true` or `false`

$$\text{Counting complexity classes} \begin{cases} \#\mathrm{P} & \equiv & ? \\ \mathrm{S{\scriptsize PAN}P} & \equiv & ? \\ \mathrm{FP} & \equiv & ? \\ \#\mathrm{L} & \equiv & ? \\ \#\mathrm{PSPACE} & \equiv & ? \\ \vdots & \vdots & \vdots \end{cases}$$

. . . but computational complexity

is not only about `true` or `false`

Counting complexity classes

$$\left\{\begin{array}{ccc} \#\mathrm{P} & \equiv & ? \\ \textsc{SpanP} & \equiv & ? \\ \mathrm{FP} & \equiv & ? \\ \#\mathrm{L} & \equiv & ? \\ \#\mathrm{PSPACE} & \equiv & ? \\ \vdots & \vdots & \vdots \end{array}\right.$$

How can we describe this **counting** classes with logic?

In this paper, we propose to use weighted logics for descriptive complexity of counting classes

We propose to use:

Quantitative Second Order Logics (QSO) = Weighted Logics over $\mathbb{N}$

Specifically, our contributions are:

1. We show that QSO **captures** many counting complexity classes.
   - $\#\mathrm{P}$, $\textsc{SpanP}$, FP, $\#\mathrm{PSPACE}$, $\textsc{MinP}$, $\textsc{MaxP}$, ...
2. We use QSO to find classes below $\#\mathrm{P}$ that has good **tractable** and **closure** properties.
3. We show how to define **quantitative recursion** over QSO in order to capture classes below FP.

Outline

Quantitative second order logic

QSO vs counting complexity

Below and beyond

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Some notation and restrictions

Given a relational signature $\mathbf{R} = \{R_1, \ldots, R_k, <\}$,
we consider **finite ordered structures** over $\mathbf{R}$ of the form:

$$\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_k^{\mathfrak{A}}, <^{\mathfrak{A}})$$

where A is the domain and $<^{\mathfrak{A}}$ is a **linear order** over A.

Let $\textsc{Struct}(\mathbf{R})$ be the set of all finite ordered structures over $\mathbf{R}$.

We consider formulas of **Second Order logic** over $\mathbf{R}$ of the form:

$$\varphi \;:=\; \texttt{True} \;\mid\; x = y \;\mid\; R(\bar{u}) \;\mid\; X(\bar{v}) \;\mid \neg\varphi \;\mid\; (\varphi \vee \varphi) \;\mid\; \exists x.\, \varphi \;\mid\; \exists X.\, \varphi$$

where $R \in \mathbf{R}$ and x and X is a first and second order variable, respectively.

The semantics of a second order formula is defined as usual.

Syntax of Quantitative Second Order logic

Definition

A QSO-formula α over **R** is given by the following **syntax**:

$$\alpha \; := \; \varphi \in \mathsf{SO} \;\mid\; s \;\mid\; (\alpha + \alpha) \;\mid\; (\alpha \cdot \alpha) \;\mid \Sigma x.\,\alpha \;\mid\; \Pi x.\,\alpha \;\mid\; \Sigma X.\,\alpha \;\mid\; \Pi X.\,\alpha$$

where φ is (boolean) second order formula and $s \in \mathbb{N}$.

Example

Let $\mathbf{R} = \{E(\cdot,\cdot), <\}$ where E encodes an edge relation.

$$\alpha \; := \; \Sigma x.\,\Sigma y.\,\Sigma z.\,\big(E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge x < y \wedge y < z\big)$$

Syntax of Quantitative Second Order logic

Definition

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where φ is (boolean) second order formula and $s \in \mathbb{N}$.

Example

Let $\mathbf{R} = \{E(\cdot,\cdot), <\}$ where E encodes an edge relation.

$$\alpha \; := \; \Sigma x.\, \Sigma y.\, \Sigma z.\, \big(\underbrace{E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge x < y \wedge y < z}_{\text{SO formula } \varphi}\big)$$

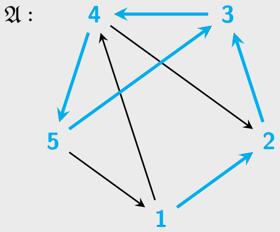
Semantics of Quantitative Second Order logic

Given a QSO-formula α, $\mathfrak{A} \in \textsc{Struct}(\mathbf{R})$ and a var. assignment $v : \mathbf{X} \to A$ we define the **semantics** $[\![\alpha]\!] : \textsc{Struct}(\mathbf{R}) \to \mathbb{N}$ recursively as follow:

$$
\begin{aligned}
[\![\varphi]\!](\mathfrak{A}, v) &= \begin{cases} 1 & \text{if } (\mathfrak{A}, v) \models \varphi \\ 0 & \text{otherwise} \end{cases} \\
[\![s]\!](\mathfrak{A}, v) &= s \\
[\![\alpha_1 + \alpha_2]\!](\mathfrak{A}, v) &= [\![\alpha_1]\!](\mathfrak{A}, v) + [\![\alpha_2]\!](\mathfrak{A}, v) \\
[\![\alpha_1 \cdot \alpha_2]\!](\mathfrak{A}, v) &= [\![\alpha_1]\!](\mathfrak{A}, v) \cdot [\![\alpha_2]\!](\mathfrak{A}, v) \\
[\![\Sigma x.\, \alpha]\!](\mathfrak{A}, v) &= \sum_{a \in A} [\![\alpha]\!](\mathfrak{A}, v[a/x]) \\
[\![\Pi x.\, \alpha]\!](\mathfrak{A}, v) &= \prod_{a \in A} [\![\alpha]\!](\mathfrak{A}, v[a/x]) \\
[\![\Sigma X.\, \alpha]\!](\mathfrak{A}, v) &= \sum_{C \subseteq A^{\text{arity}(X)}} [\![\alpha]\!](\mathfrak{A}, v[C/X]) \\
[\![\Pi X.\, \alpha]\!](\mathfrak{A}, v) &= \prod_{C \subseteq A^{\text{arity}(X)}} [\![\alpha]\!](\mathfrak{A}, v[C/X])
\end{aligned}
$$

Semantics of Quantitative Second Order logic

Example (counting the triangles in a graph)



$$\text{triangle}(x, y, z) \;:=\; E(x, y) \wedge E(y, z) \wedge E(z, x) \wedge x < y \wedge y < z$$

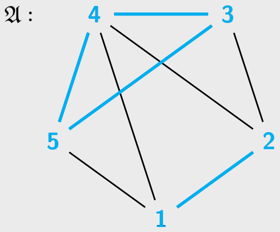
$$[\![\text{triangle}]\!](\mathfrak{A}, 3, 4, 5) = 1 \qquad [\![\text{triangle}]\!](\mathfrak{A}, 1, 2, 3) = 0$$

$$\alpha \;:=\; \Sigma x. \Sigma y. \Sigma z.\ \text{triangle}(x, y, z)$$

$$[\![\alpha]\!](\mathfrak{A}) = 3$$

Semantics of Quantitative Second Order logic

Example (counting the number of cliques in a graph)



$$\text{clique}(X) := \forall x.\, \forall y.\, (X(x) \wedge X(y) \wedge x \neq y) \rightarrow E(x, y)$$

$$[\![\text{clique}]\!](\mathfrak{A}, \{3, 4, 5\}) = 1 \qquad [\![\text{clique}]\!](\mathfrak{A}, \{1, 2\}) = 1$$

$$\alpha := \Sigma X.\, \text{clique}(X)$$

$$[\![\alpha]\!](\mathfrak{A}) = 18$$

Subfragments and extentions of QSO

$$\alpha \ := \ \varphi \in \mathsf{SO} \ | \ s \ | \ (\alpha + \alpha) \ | \ (\alpha \cdot \alpha) \ | \, \Sigma x.\, \alpha \ | \ \Pi x.\, \alpha \ | \ \Sigma X.\, \alpha \ | \ \Pi X.\, \alpha$$

$$\mathsf{QSO} \ = \ \underbrace{\mathsf{QSO}}_{\alpha}(\overbrace{\mathsf{SO}}^{\varphi})$$

We can restrict or extend the language of φ:

$$\begin{aligned} \mathsf{QSO(FO)} &\quad := \quad \varphi \text{ is restricted to } \textbf{FO logic}. \\ \mathsf{QSO(LFP)} &\quad := \quad \varphi \text{ is restricted to } \textbf{LFP logic}. \end{aligned}$$

We can restrict or extend the language of α:

$$\begin{aligned} \mathsf{QFO(SO)} &\quad := \quad \alpha \text{ is restricted to } \textbf{first order operators} \text{ (i.e. } s, +, \Sigma x., \Pi x.\text{)}. \\ \Sigma\mathsf{QSO(SO)} &\quad := \quad \alpha \text{ is restricted to } \textbf{sum operators} \text{ (i.e. } s, +, \Sigma x., \Sigma X.\text{)} \end{aligned}$$

Or both φ and α:

$$\begin{aligned} \mathsf{QFO(LFP)} \quad = \quad & \alpha \text{ is restricted to } \textbf{first order operators} \\ & \text{and } \varphi \text{ is restricted to } \textbf{LFP logic}. \end{aligned}$$

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Capturing a counting complexity class with QSO

- Recall that a counting complexity $\mathcal{C} \subseteq \{f : \Sigma^* \to \mathbb{N}\}$.
- Let $\mathrm{enc}(\mathfrak{A})$ be any reasonable encoding of $\mathfrak{A}$ into a string in Σ^*.

Definition

Let $\mathcal{F}$ be a fragment or extention of QSO and $\mathcal{C}$ a counting complexity class. Then $\mathcal{F}$ **captures** $\mathcal{C}$ over ordered **R**-structures if:

1. for every $\alpha \in \mathcal{F}$, there exists $f \in \mathcal{C}$ such that $[\![\alpha]\!](\mathfrak{A}) = f(\mathrm{enc}(\mathcal{A}))$ for every $\mathfrak{A} \in \textsc{Struct}[\mathbf{R}]$.
2. for every $f \in \mathcal{C}$, there exists $\alpha \in \mathcal{F}$ such that $f(\mathrm{enc}(\mathcal{A})) = [\![\alpha]\!](\mathfrak{A})$ for every $\mathfrak{A} \in \textsc{Struct}[\mathbf{R}]$.

$\mathcal{F}$ **captures** $\mathcal{C}$ over ordered structures if $\mathcal{F}$ captures $\mathcal{C}$ over ordered **R**-structures **for every signature** **R**.

What counting classes can be captured by QSO?

Counting complexity classes

$$\left\{\begin{array}{ccc} \#\mathrm{P} & \equiv & ? \\ \textsc{SpanP} & \equiv & ? \\ \mathrm{FP} & \equiv & ? \\ \#\mathrm{L} & \equiv & ? \\ \#\mathrm{PSPACE} & \equiv & ? \\ \vdots & \vdots & \vdots \end{array}\right.$$

We show that most of these classes
can be captured by subfragments or extentions of QSO

How to capture $\#\mathrm{P}$?

$f \in \#\mathrm{P}$ iff there exists an **NP machine** M
such that $f(x) = \#\mathrm{accepts}_M(x)$ for all $x \in \Sigma^*$.

$\Sigma\mathrm{QSO(FO)}$:= α restricted to **sum operators** (i.e. $s, +, \Sigma x., \Sigma X.$)
and φ restricted to **FO logic**.

Theorem

$\Sigma\mathrm{QSO(FO)}$ captures $\#\mathrm{P}$ over ordered structures.

How to capture SPANP?

$$\#\mathrm{P} \quad \equiv \quad \Sigma\mathsf{QSO}(\mathsf{FO})$$

$f \in$ SPANP iff there exists an **NP machine** M with **output** such that $f(x) = \#\mathrm{outputs}_M(x)$ for all $x \in \Sigma^*$.

$\Sigma\mathsf{QSO}(\exists\mathsf{SO})$:= α restricted to **sum operators** (i.e. $s, +, \Sigma x., \Sigma X.$) and φ restricted to **existential SO logic**.

Theorem

$\Sigma\mathsf{QSO}(\exists\mathsf{SO})$ captures SPANP over ordered structures.

#P and SPANP were shown to be captured by a different framework of Saluja et al. and Compton et al.

How to capture FP?

$$\begin{array}{rcl} \#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\ \textsc{SpanP} & \equiv & \Sigma\mathsf{QSO}(\exists\mathsf{SO}) \end{array}$$

$f \in \mathrm{FP}$ iff there exists **PTIME machine** M with output such that $f(x) = M(x)$ for all $x \in \Sigma^*$.

QFO(LFP) := α restricted to **first order op.** (i.e. $+, \cdot, \Sigma x., \Pi x.$) and φ restricted to **LFP logic**.

Theorem

QFO(LFP) captures FP over ordered structures.

How to capture FPSPACE?

$$\begin{array}{ccc} \#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\ \textsc{SpanP} & \equiv & \Sigma\mathsf{QSO}(\exists\mathsf{SO}) \\ \#\mathrm{P} & \equiv & \mathsf{QFO}(\mathsf{LFP}) \end{array}$$

$f \in \mathrm{FPSPACE}$ iff there exists **PSPACE machine** M with output such that $f(x) = M(x)$ for all $x \in \Sigma^*$.

QSO(PFP) := φ restricted to **PFP logic**.

Theorem

QSO(PFP) captures FPSPACE over ordered structures.

How to capture $\mathrm{FPSPACE}(\text{poly})$?

$$\begin{array}{ccc} \#\mathrm{P} & \equiv & \Sigma\mathsf{QSO}(\mathsf{FO}) \\ \textsc{SpanP} & \equiv & \Sigma\mathsf{QSO}(\exists\mathsf{SO}) \\ \#\mathrm{P} & \equiv & \mathsf{QFO}(\mathsf{LFP}) \\ \mathrm{FPSPACE} & \equiv & \mathsf{QSO}(\mathsf{PFP}) \end{array}$$

$f \in \mathrm{FPSPACE}(\text{poly})$ iff there exists **PSPACE machine** M with output of polynomial size such that $f(x) = M(x)$ for all $x \in \Sigma^*$.

$\mathsf{QFO}(\mathsf{PFP})$:= α restricted to **first order op.** (i.e. $+, \cdot, \Sigma x., \Pi x.$) and φ restricted to **PFP logic**.

Theorem

$\mathsf{QFO}(\mathsf{PFP})$ captures $\mathrm{FPSPACE}(\text{poly})$ over ordered structures.

More classes?

$\#\mathrm{P}$	$\equiv$	$\Sigma\mathsf{QSO}(\mathsf{FO})$
$\textsc{SpanP}$	$\equiv$	$\Sigma\mathsf{QSO}(\exists\mathsf{SO})$
$\#\mathrm{P}$	$\equiv$	$\mathsf{QFO}(\mathsf{LFP})$
$\mathrm{FPSPACE}$	$\equiv$	$\mathsf{QSO}(\mathsf{PFP})$
$\mathrm{FPSPACE}(\mathsf{poly})$	$\equiv$	$\mathsf{QFO}(\mathsf{PFP})$
$\textsc{GapP}$	$\equiv$	$\Sigma\mathsf{QSO}_{\mathbb{Z}}(\mathsf{FO})$
$\textsc{MaxP}$	$\equiv$	$\mathsf{MaxQSO}(\mathsf{FO})$
$\textsc{MinP}$	$\equiv$	$\mathsf{MinQSO}(\mathsf{FO})$

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Use QSO to understand classes **below** $\#\mathrm{P}$

$$\#\mathrm{P} \quad \equiv \quad \Sigma\mathsf{QSO}(\mathsf{FO})$$

We consider subfragments below FO:

$$\begin{aligned}
\Sigma_0 &= \{\ \theta \in \mathsf{FO} \mid \theta \text{ has no first-order quantifiers } \} \\
\Sigma_1 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \exists \bar{x}.\, \theta(\bar{x}) \ \wedge\ \theta \in \Sigma_0\ \} \\
\Pi_1 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \forall \bar{x}.\, \theta(\bar{x}) \ \wedge\ \theta \in \Sigma_0\ \} \\
\Sigma_2 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \exists \bar{x}.\, \forall \bar{y}.\, \theta(\bar{x}, \bar{y}) \ \wedge\ \theta \in \Sigma_0\ \} \\
\Pi_2 &= \{\ \varphi \in \mathsf{FO} \mid \varphi = \forall \bar{x}.\, \exists \bar{y}.\, \theta(\bar{x}, \bar{y}) \ \wedge\ \theta \in \Sigma_0\ \}
\end{aligned}$$

Use QSO to understand classes **below** $\#\mathrm{P}$

$$\#\mathrm{P} \quad \equiv \quad \Sigma\mathrm{QSO}(\mathrm{FO})$$

Saluja et. al. counting classes below $\#\mathrm{P}$

$$\#\Sigma_0 \subsetneq \#\Sigma_1 \subsetneq \#\Pi_1 \subsetneq \#\Sigma_2 \subsetneq \#\Pi_2 = \#\mathrm{FO} \equiv \#\mathrm{P}$$

Theorem (ΣQSO-hierarchy)

$$\#\Sigma_1$$

$$\subsetneq \qquad \subsetneq$$

$$\#\Sigma_0 \qquad \Sigma\mathrm{QSO}(\Sigma_1) \subsetneq \#\Pi_1 = \Sigma\mathrm{QSO}(\Pi_1) \subsetneq \#\Sigma_2$$

$$\subsetneq \qquad \subsetneq$$

$$\Sigma\mathrm{QSO}(\Sigma_0)$$

$$\#\Sigma_2 = \Sigma\mathrm{QSO}(\Sigma_2) \subsetneq \#\Pi_2 = \Sigma\mathrm{QSO}(\Pi_2) \equiv \#\mathrm{P}$$

Use QSO to understand classes **below** $\#\mathrm{P}$

Theorem (ΣQSO-hierarchy)

$$\#\Sigma_0 \subsetneq \#\Sigma_1 \subsetneq \Sigma QSO(\Sigma_1)$$

$$\#\Sigma_0 \subsetneq \Sigma QSO(\Sigma_0) \subsetneq \Sigma QSO(\Sigma_1)$$

$$\Sigma QSO(\Sigma_1) \subsetneq \#\Pi_1 = \Sigma QSO(\Pi_1) \subsetneq \#\Sigma_2$$

$$\#\Sigma_2 = \Sigma QSO(\Sigma_2) \subsetneq \#\Pi_2 = \Sigma QSO(\Pi_2) \equiv \#\mathrm{P}$$

Theorem (good decision and closure properties)

The class $\Sigma QSO(\Sigma_1[FO])$ is closed under sum, multiplication and **subtraction by one**. Moreover, $\Sigma QSO(\Sigma_1[FO]) \subseteq \mathrm{TotP}$ and every function in $\Sigma QSO(\Sigma_1[FO])$ has an FPRAS.

Substraction by one is the most technical result of the paper.

Extend QSO to capture complexity classes **beyond** QSO

We extend QFO with **recursion**:

RQFO	$=$	QFO with **quantitative** recursion.
TQFO	$=$	QFO with **quantitative** transitive closure.

Theorem

1. RQFO(FO) captures FP over the class of ordered structures.
2. TQFO(FO) captures $\#\mathrm{L}$ over the class of ordered structures.

Conclusions and future work

"We believe that **quantitative logics** *are the right framework for Descriptive complexity of* **counting complexity classes***."*

Plenty of open problems here ...

1. **Logical characterization** of classes like $\textsc{TotP}, \textsc{SpanL}, \ldots$
2. **Compl. characterization** of subfragments like $\mathsf{QSO(FO)}, \mathsf{QFO(FO)}, \ldots$
3. Use quantitative logic to find complexity **classes with good properties**.
4. Understand the **expressiveness** of QSO and their subfragments.

Thanks! Questions?